Optimization Theory

Applied Mathematics

Final Exam (version 1.0)

Theoretical Questions (2 points each)

1. How to check that a real-valued twice continuously differentiable function is convex?
2. Can steepest descent method with optimal stepsize applied to a twice continuously differentiable function converge to a point which is not a local minimum? If yes, what can be said about such a point?
3. When is the first phase of the simplex method necessary? How can it be seen in the simplex tableau?
4. Suppose that the solution to the dual program is $[0, 4, 3, 1]$. If paying \$2 we can increase each constraint in the primal program by 1, increasing which constraints may be profitable (given that the objective function is also denominated in dollars)?
5. Suppose you have an integer program with a large number of constraints. Why may it make sense to solve a corresponding linear program instead and using its solution find an approximately optimal solution to the integer program rather than solve the integer program explicitly?

Problems (6 points each)

6. The steepest descent method with optimal stepsize is applied to solve the problem
$$\text{minimize } f(x,y) = 4x^2 - 4xy + 2y^2.$$
Show by induction that for $(x_0, y_0) = (2,3)$, $(x_{2k}, y_{2k}) = \frac{1}{5^k}(2,3)$. Deduce the minimizer of $f(x,y)$ using the obtained sequence of approximations.
7. The factory produces 2 products using 3 machines. It takes 1.2hrs on 1st machine and 0.8hrs on the second machine to produce a unit of product A. It takes 0.5hr on 1st machine, 0.7hrs on the 2nd one and 1.4hrs on the 3rd machine to produce a unit of product B. Each machine may operate at most for 40 hours per week. Weekly demand for product A is 20 units while that for product B is 40 units. The profit from selling a unit of product A is \$12 while that from selling a unit of product B is \$8. Write (without solving) a linear program maximizing profit of the factory subject to all the production and demand constraints. Write also its matrix form.
8. Solve the linear program
$$\begin{array}{ll} \text{Maximize} & x_1 + 3x_2 \\ \text{subject to} & x_1 + x_2 \le 4, \\ & 2x_1 + x_2 \le 6, \\ & x_1 \ge 0, x_2 \ge 0. \end{array}$$
using the simpex method. Make a graph of the feasible set and indicate on it which vertices are visited by the simplex procedure in its successive steps.
9. Write the problem
$$\begin{array}{ll} \text{Maximize} & x_1 - 3x_2 \\ \text{subject to} & ((x_1 + x_2 > 4) \vee (5x_1 + x_2 > 5)) \Longrightarrow (x_1 + 2x_2 \le 2) \\ & x_1, x_2 \ge 0 \end{array}$$
as an integer program.
10. Find the distance between the circle $x^2 + y^2 = 1$ and the hyperbola $xy = 4$ using the method of Lagrange multipliers.

Good luck!

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